Unit - III

Recurrence Relations

A recurrence relation for the sequence $\{a_n\}$ is an equations that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integers $n \geq 1$.

Note:- Recurrence relation is also called difference Equation.

Eg:- 1. If S_n denotes the sum of the first 'n' +ve integers, then $S_n = n + S_{n-1}$ which is a recurrence relation.

2. If S_n denotes the nth term of a geometric progression with common ratio 'r' then $S_n = rS_{n-1}$ which is a recurrence relation.

3. $a_n - 3a_{n-1} + 2a_{n-2} = 0$

4. $a_n - 5a_{n-1} + 6a_{n-2} = n^2 + 1$

5. $a_n^2 + a_{n-1}^2 = -1$

6. $a_n - (n-1)a_{n-1} + (n-1)a_{n-2} = 0$

Definition:- Suppose n & k are non-ve integers. A recurrence relation of the form $C_0(n)\, a_n + C_1(n) a_{n-1} + \cdots + C_k(n)\, a_{n-k} = f(n)$ —(1)

for $n \geq k$, where $C_0(n), C_1(n) \cdots C_k(n)$ & $f(n)$ are solutions of 'n' is said to be linear recurrence relation.

If $c_0(n), c_1(n) \ldots c_k(n)$ are constants, then the recurrence relation ① is known as a linear recurrence relation with constant coefficients.

If $f(n)=0$ then ① is said to be homogeneous recurrence relation.

If $f(n) \neq 0$ then ① is said to be non-homogeneous recurrence relation.

Note. All Examples above are linear recurrence relations Except ⑤ because it contains square term.

The relations ①, ② have degree '1', ③, ④ have degree 2.

The relations 2, 3, 6 are homogeneous.

Relations 1, 4 are non-homogeneous.

The relations 1, 2, 3, 4 are linear with constant coefficients

Solutions of Recurrence Relations.

A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of a recurrence relation if each value a_n i.e $a_0, a_1, a_2, \ldots a_n \ldots$ statistics the recurrence relation.

Example. $\{a_n\}_{n=0}^{\infty}$ where $a_n = 2^n$ is the solution of the recurrence relation $a_n = 2a_{n-1}, n \geq 1$ and the sequence $\{c2^n\}_{n=0}^{\infty}$ where c is constant is also solution of recurrence relation $a_n = 2a_{n-1}, n \geq 1$

Solving Recurrence Relations by substitution & Generating functions

There are 3 methods of solving recurrence relations

1) Substitution (Iteration)
2) Generating functions
3) characteristic roots.

Method I (Substitution Method)

In this method, the recurrence relation for a_n is used repeated to solve for a general expression for a_n in terms of n.

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by substitution method.

$n=1 \Rightarrow$ $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$\vdots$

$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + f(3) + \cdots + f(n)$

$= a_0 + \sum_{k=1}^{n} f(k).$

2. Solve the recurrence relation $a_n = a_{n-1} + n^2$ where $a_0 = 7$ by substitution method.

Initial Condition $a_0 = 7$

$$\cancel{a_1 = a_0 +}$$

$$a_n = a_{n-1} + n^2.$$

$$a_1 = a_0 + 1^2 = 7 + 1^2 \quad [a_0 = 7]$$

$$a_2 = a_1 + 2^2 = 7 + 1^2 + 2^2$$

$$a_3 = a_2 + 3^2 = 7 + 1^2 + 2^2 + 3^2$$

$$\vdots$$

$$a_n = 7 + (1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= 7 + \frac{n(n+1)(2n+1)}{6}$$

which is required solution.

3. Solve the following recurrence relations by substitution

1. $a_n = a_{n-1} + \frac{1}{n(n+1)}$ where $a_0 = 1$.

Initial Condition is $a_0 = 1$

$$a_1 = a_0 + \frac{1}{1.2} = 1 + \frac{1}{1.2}$$

$$a_2 = a_1 + \frac{1}{2.3} = 1 + \frac{1}{1.2} + \frac{1}{2.3}$$

$$a_3 = a_2 + \frac{1}{3.4} = 1 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

$$\vdots$$

$$a_n = 1 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}$$

$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 + 1 - \frac{1}{n+1}$

$= 2 - \frac{1}{n+1}$ is the solution.

(ii) $a_n = a_{n-1} + 3^n$ where $a_0 = 1$

Initial Condition is $a_0 = 1$

$a_1 = a_0 + 3^1 = 1 + 3$

$a_2 = a_1 + 3^2 = 1 + 3 + 3^2$

$a_3 = a_2 + 3^2 = 1 + 3 + 3^2 + 3^3$

$\vdots$

$a_n = 1 + 3 + 3^2 + 3^3 + \cdots + 3^n$

$a_n = \frac{3^{n+1} - 1}{3 - 1}$

$a_n = \frac{3^{n+1} - 1}{2}$ is the solution.

$a_n = \frac{a(r^{n+1} - 1)}{r - 1}$

which is in G.P

$r = 3 > 1$ $(r > 1)$

$\boxed{r < 1 \Rightarrow \frac{1 - r^{n+1}}{1 - r}}$

(iii) $a_n = a_{n-1} + n\,3^n$ given $a_0 = 1$

$a_n = a_{n-1} + n\,3^n$, $a_0 = 1$

$a_1 = a_0 + 1 \cdot 3^1 = 1 + 1 \cdot 3$

$a_2 = a_1 + 2 \cdot 3^2 = 1 + 1 \cdot 3 + 2 \cdot 3^2$

$a_3 = a_2 + 3 \cdot 3^3 = 1 + 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots$

$\vdots$

$a_n = 1 + 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n.$

$a_n = 1 + \sum_{r=1}^{\infty} r \cdot 3^r$ is the required solution.

(iv) Solve the recurrence relation $a_n = a_{n-1} + \frac{n(n+1)}{2}$, $n \geq 1$

$$a_n = a_{n-1} + \frac{n(n+1)}{2}$$

$$a_1 = a_0 + 1.1 \quad = a_0 + 1$$

$$a_2 = a_1 + 3 \quad = a_0 + 1 + 3$$

$$a_3 = a_2 + 6 \quad = a_0 + 1 + 3 + 6$$

$$\vdots$$

$$a_n = a_{n-1} + \frac{n(n+1)}{2}$$

$a_n = a_0 + 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}$ is the required solution

Method 2 Generating Functions

Recurrence relations can also be solved by using generating functions. Some Equivalent Expressions used are given below

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - \cdots - a_{k-1} x^{k-1}$$

$$\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[A(x) - a_0 - a_1 x - \cdots a_{k-2} x^{k-2}\right]$$

$$\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[A(x) - a_0 - a_1 x - \cdots - a_{k-3} x^{k-3}\right]$$

$$\sum_{n=k}^{\infty} a_{n-3} x^n = x^3\left[A(x) - a_0 - a_1 x - \cdots - a_{k-4} x^{k-4}\right]$$

$$\vdots$$

$$\sum_{n=k}^{\infty} a_{n-k} x^n = x^k \left[A(x)\right]$$

where $A(x)$ is called a generating function for a given recurrence relations.

Prob 1

1. Solve the recurrence relation: $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$

1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

2. Next Multiply each term in the recurrence relation by x^n & sum from 2 to ∞.

$$\underbrace{\sum_{n=2}^{\infty} a_n x^n}_{a_2x^2} - 7\underbrace{\sum_{n=2}^{\infty} a_{n-1} x^n}_{a_1x^2} + 10\underbrace{\sum_{n=2}^{\infty} a_{n-2} x^n}_{a_0x^2} = 0$$

(3) Replace each infinite sum by an Expression from the Equivalent Expressions.

$$[A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 10x^2[A(x)] = 0$$

$$A(x)[1 - 7x + 10x^2] = a_0 + a_1 x - 7a_0 x.$$

$$A(x) = \frac{a_0 + a_1 x - 7a_0 x}{1 - 7x + 10x^2}$$

$$= \frac{a_0 + x(a_1 - 7a_0)}{(1-2x)(1-5x)}$$

Decompose $A(x)$ as a Sum of Partial Fractions.

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-5x}.$$

Where c_1 & c_2 are Constants, as yet undetermined.

Express $A(x)$ as a Sum of finite Series

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-5x}.$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 5^n x^n$$

Express a_n as the Coefficient of x^n in $A(x)$ & in the Sum of the other Series $a_n = c_1 2^n + c_2 5^n$.

Prob 2

Find a general Expression for a_n using Generating Function to $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $n \geq 2$

Sol Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$a_n - 7a_{n-1} + 12a_{n-2} = 0, \quad n \geq 2 \quad \text{——— (1)}$$

Multiply Each term (1) by x^n & Sum from 2 to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 12 \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$[A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 12x^2[A(x)] = 0$$

$$A(x)[1 - 7x + 12x^2] - a_0 + 7x a_0 - a_1 x = 0$$

$$A(x)[1 - 7x + 12x^2] = a_0 - 7x a_0 + a_1 x$$

$$A(x) = \frac{a_0 + x(a_1 - 7a_0)}{1 - 7x + 12x^2}$$

$$= \frac{a_0 + x(a_1 - 7a_0)}{(1-3x)(1-4x)}$$

$$= \frac{c_1}{1-3x} + \frac{c_2}{1-4x}.$$

$$A(x) = c_1 \sum_{n=0}^{\infty} 3^n x^n + c_2 \sum_{n=0}^{\infty} 4^n x^n$$

$$= c_1 3^n + c_2 4^n.$$

Prob 3 Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0, \quad n \geq 2$

Sol:- Let $A(x) = \sum_{n=0}^{\infty} a_n x^n.$

Multiply each term by x^n & sum from 2 to ∞

$$\sum_{n=2}^{\infty} x^n a_n - 5 \sum_{n=2}^{\infty} a_{n-1} x^n + 6 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[A(x) - a_0 - a_1 x] - 5x[A(x) - a_0] + 6x^2[A(x)] = 0$$

$$A(x)[1 - 5x + 6x^2] - a_0 - a_1 x + 5x a_0 = 0$$

$$A(x)[1 - 5x + 6x^2] = a_0 + a_1 x - 5x a_0$$

$$A(x) = \frac{a_0 + a_1 x - 5x a_0}{1 - 5x + 6x^2}$$

$$= \frac{a_0 + x(a_1 - 5a_0)}{(3x-1)(2x-1)}$$

$$A(x) = \frac{c_1}{1-3x} + \frac{c_2}{1-2x}$$

$$= c_1 \sum_{n=0}^{\infty} 3^n x^n + c_2 \sum_{n=0}^{\infty} 2^n x^n.$$

$$= c_1 3^n + c_2 2^n.$$

$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 5^n x^n$

Express a_n as the Coefficient of x^n in $A(x)$ & in the Sum of the other Series $a_n = c_1 2^n + c_2 5^n$.

Prob 2

Find a general Expression for a_n using Generating Function to $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $n \geq 2$

Sol Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$a_n - 7a_{n-1} + 12a_{n-2} = 0, \; n \geq 2 \quad \text{——— (1)}$$

Multiply Each term (1) by x^n & Sum from 2 to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 12\sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$[A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 12x^2[A(x)] = 0$$

$$A(x)[1 - 7x + 12x^2] - a_0 + 7x a_0 - a_1 x = 0$$

$$A(x)[1 - 7x + 12x^2] = a_0 - 7x a_0 + a_1 x$$

$$A(x) = \frac{a_0 + x(a_1 - 7a_0)}{1 - 7x + 12x^2}$$

$$= \frac{a_0 + x(a_1 - 7a_0)}{(1-3x)(1-4x)}$$

$$= \frac{c_1}{1-3x} + \frac{c_2}{1-4x}.$$

$$A(x) = C_1 \sum_{n=0}^{\infty} 3^n x^n + C_2 \sum_{n=0}^{\infty} 4^n x^n$$

$$= C_1 3^n + C_2 4^n.$$

Prob 3 Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0, \; n \geq 2$

Sol:- Let $A(x) = \sum_{n=0}^{\infty} a_n x^n.$

Multiply each term by x^n & sum from 2 to ∞

$$\sum_{n=2}^{\infty} x^n a_n - 5 \sum_{n=2}^{\infty} a_{n-1} x^n + 6 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[A(x) - a_0 - a_1 x] - 5x[A(x) - a_0] + 6x^2 [A(x)] = 0$$

$$A(x)[1 - 5x + 6x^2] - a_0 - a_1 x + 5x a_0 = 0$$

$$A(x)[1 - 5x + 6x^2] = a_0 + a_1 x - 5x a_0$$

$$A(x) = \frac{a_0 + a_1 x - 5x a_0}{1 - 5x + 6x^2}$$

$$= \frac{a_0 + x(a_1 - 5a_0)}{(3x-1)(2x-1)}$$

$$A(x) = \frac{C_1}{1-3x} + \frac{C_2}{1-2x}$$

$$= C_1 \sum_{n=0}^{\infty} 3^n x^n + C_2 \sum_{n=0}^{\infty} 2^n x^n.$$

$$= C_1 3^n + C_2 2^n.$$

Prob 4 Solve $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$ by generating function.

Sol: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

Multiply each term by x^n & sum

$$\sum_{n=3}^{\infty} a_n x^n - 6\sum_{n=3}^{\infty} a_{n-1}x^n + 12\sum_{n=3}^{\infty} a_{n-2}x^n - 8\sum_{n=3}^{\infty} a_{n-3} = 0$$

$$\left[A(x) - a_0 - a_1x - a_2x^2\right] - 6x\left[A(x) - a_0 - a_1x\right] + 12x^2\left[A(x) - a_0\right]$$
$$- 8x^3 A(x) = 0$$

$$A(x)\left[1 - 6x + 12x^2 - 8x^3\right] - a_0 - a_1x - a_2x^2 + 6xa_0 + 6a_1x^2 - 12a_0x^2 = 0$$

$$\cancel{A(x) = \frac{a_0 + a_1x - 5xa_0}{1 - 6x + 12x^2 - 8x^3}}$$

$$A(x)\left[1 - 6x + 12x^2 - 8x^3\right] = a_0 + a_1x + a_2x^2 - 6xa_0 - 6a_1x^2 + 12a_0x^2$$

$$A(x) = \frac{a_0 + a_1x - 6a_0x + a_2x^2 - 6a_1x^2 + 12a_0x^2}{1 - 6x + 12x^2 - 8x^3}$$

$$= \frac{a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 12a_0)x^2}{(1-2x)^3}$$

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{(1-2x)^2} + \frac{c_3}{(1-2x)^3}$$

$$= c_1(1-2x)^{-1} + c_2(1-2x)^{-2} + c_3(1-2x)^{-3}$$

$$= c_1 \sum_{n=0}^{\infty} (2x)^n + c_2 \sum_{n=0}^{\infty} c(n+1, n)\,(2x)^n + \sum_{n=0}^{\infty} c_3\, c(n+2, n)(2x)^n$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} c_2 (n+1)\, 2^n x^n + \sum_{n=0}^{\infty} c_3 \frac{(n+2)(n+1)}{2}\, 2^n x^n.$$

$$= \sum_{n=0}^{\infty} \left[c_1 2^n + c_2 (n+1)\, 2^n + c_3 \frac{(n+2)(n+1)}{2} \cdot 2^n \right] x^n.$$

$$a_n = c_1 2^n + c_2 (n+1)\, 2^n + c_3 \frac{(n+2)(n+1)}{2} \cdot 2^n.$$

Prob Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ for $n \geq 3$

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$$

Sol :- Let $A(x) = \sum_{n=0}^{\infty} a_n x^n.$

Multiply each term by x^n & sum from 3 to ∞.

$$\sum_{n=3}^{\infty} a_n x^n - 8 \sum_{n=3}^{\infty} a_{n-1} x^n + 21 \sum_{n=3}^{\infty} a_{n-2} x^n - 18 \sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

$$\left[A(x) - a_0 - a_1 x - a_2 x^2\right] - 8x\left[A(x) - a_0 - a_1 x\right] + 21x^2\left[A(x) - a_0\right]$$
$$- 18x^3\left[A(x)\right] = 0$$

$$A(x)\left[1 - 8x + 21x^2 - 18x^3\right] = a_0 + a_1 x + a_2 x^2 - 8x a_0 + 8x^2 a_1 + 21x^2 a_0$$

$$A(x) = \frac{a_0 + x(a_1 - 8x) + x^2(a_2 + 8a_1 + 21a_0)}{(1-2x)(1-3x)^2}$$

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{(1-3x)^2}$$

$$= c_1(1-2x)^{-1} + c_2(1-3x)^{-1} + c_3(1-3x)^{-2}$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 3^n x^n + c_3 \sum_{n=0}^{\infty} (n+1, n) 3^n x^n$$

$$= \sum_{n=0}^{\infty} c_1 2^n x^n + \sum_{n=0}^{\infty} c_2 3^n x^n + \sum_{n=0}^{\infty} c_3 (n+1) 3^n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n \left[c_1 2^n + c_2 3^n + (c_3 n+1) 3^n \right]$$

$$a_n = c_1 2^n + c_2 3^n + (n+1) 3^n.$$

Prob

Solve the recurrence relation $a_n = 3a_{n-1} + 2$, $n \geq 1$, & $a_0 = 1$ using Generating Fraction.

Sol

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n.$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n.$$

$$A(x) - a_0 = 3A(x)x + 2\left[x + x^2 + x^3 + \dots\right]$$

$$Ax(1-3x) = 1 + 2x(1 + x + x^2 + \dots)$$

$$= 1 + 2x(1-x)^{-1}$$

$$= 1 + \frac{2x}{1-x}$$

$$= \frac{1-x+2x}{1-x} = \frac{1+x}{1-x}.$$

$$A(x) = \frac{1+x}{(1-x)(1+3x)} = \frac{A}{1-x} + \frac{B}{1-3x}.$$

$A = -1, \quad B = 2.$

$$\sum a_n x^n = \frac{-1}{1-x} + \frac{2}{1-3x}$$

$$= -1 \sum_{n=0}^{\infty} x^n + 2 \sum_{n=0}^{\infty} (3x)^n.$$

$$= -1 \sum_{n=0}^{\infty} x^n + 2 \sum_{n=0}^{\infty} 3^n x^n$$

$$a_n = -1 + 2.3^n.$$

Prob Solve the recurrence relation $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$

Generating function $a_0 = -3$, $a_1 = -10$

Sol Given recurrence relation is

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n.$

Multiply each term by x^n & sum from 2 to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[A(x) - a_0 - a_1 x] - 9x[A(x) - a_0] + 20x^2[A(x)] = 0$$

$$A(x)[1 - 9x + 20x^2] - a_0 - a_1 x + 9x a_0 = 0$$

$$A(x)[1 - 9x + 2x^2] = a_0 + a_1 x - 9x a_0$$

$$A(x) = \frac{a_0 + a_1 x - 9x a_0}{1 - 9x + 2x^2}$$

Given $a_0 = -3$, $a_1 = -10$.

$$A(x) = \frac{-3 - 10x - 9x(-3)}{1 - 9x + 20x^2}$$

$$A(x) = \frac{17x - 3}{(1-4x)(1-5x)}$$

$$\frac{17x-3}{(1-4x)(1-5x)} = \frac{A}{1-4x} + \frac{B}{1-5x}.$$

$$A + B = -3$$
$$A + 2 = -3$$
$$\boxed{A = -5}$$

$$-5A - 4B = 17$$
$$5A + 5B = -15$$
$$\underline{B = 2}$$

$$17x - 3 = A(1-5x) + B(1-4x)$$

$$= \frac{-5}{1-4x} + \frac{2}{1-5x}.$$

$$= -5(1-4x)^{-1} + 2(1-5x)^{-1}$$

$$A(x) = -5\sum_{n=0}^{\infty}(4x)^n + 2\sum_{n=0}^{\infty}(5x)^n$$

$$\sum_{n=0}^{\infty} a_n x^n = -5\sum_{n=0}^{\infty} 4^n x^n + 2\sum_{n=0}^{\infty} 5^n x^n$$

$$= \sum_{n=0}^{\infty}\left[-5 \cdot 4^n + 2 \cdot 5^n\right]x^n$$

$$a_n = -5 \cdot 4^n + 2 \cdot 5^n.$$

Prob Solve the recurrence relation $a_n - 6a_{n-1} = 0$ for $n \geq 1$. and $a_0 = 1$ by using G.F (Generating Function).

Sol. Given recurrence relation is $a_n - 6a_{n-1} = 0$ ①

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the Generating function

Multiply each term in ① by x^n & sum from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n - 6 \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$[A(x) - a_0] - 6x[A(x)] = 0$$

$$A(x)[1 - 6x] = a_0$$

$$A(x) = \frac{a_0}{1-6x}$$

$a_0 = 1$.

$$\sum_{n=0}^{\infty} a_n x^n = A(x) = \frac{1}{1-6x} = (1-6x)^{-1}$$

$$= \sum_{n=0}^{\infty} (6x)^n$$

$$= \sum_{n=0}^{\infty} 6^n x^n.$$

$\therefore a_n = 6^n$ which is required solution.

Prob Solve the recurrence relation $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$

for $n \geq 3$ with initial Conditions $a_0 = 0, a_1 = 1, a_2 = 10$

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0 \quad \text{—— (1)}$$

Sol:- Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

Multiply each term in (1) by x^n & Sum from 3 to ∞

$$\sum_{n=3}^{\infty} a_n x^n - 9\sum_{n=3}^{\infty} a_{n-1}x^n + 26\sum_{n=3}^{\infty} a_{n-2}x^n - 24\sum_{n=3}^{\infty} a_{n-3}x^n = 0$$

$$\left[A(x) - a_0 - a_1 x - a_2 x^2\right] - 9x\left[A(x) - a_0 - a_1 x\right] + 26x^2\left[A(x) - a_0\right] - 24x^3 A(x) = 0$$

$$A(x)\left[1 - 9x + 26x^2 - 24x^3\right] - a_0 - a_1 x - a_2 x^2 + 9xa_0 + 9a_1 x^2 - 26x^2 a_0 = 0$$

$$A(x)\left[1 - 9x + 26x^2 - 24x^3\right] - 0 - x - 10x^2 + 9x^2 - 0 = 0$$

$$A(x)\left[1 - 9x + 26x^2 - 24x^3\right] = x + x^2$$

$$A(x) = \frac{x + x^2}{1 - 9x + 26x^2 - 24x^3}$$

$$= \frac{x + x^2}{(1-2x)(1-3x)(1-4x)}$$

$$\frac{x+x^2}{(1-2x)(1-3x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-4x}.$$

$$x+x^2 = A(1-3x)(1-4x) + B(1-2x)(1-4x) + C(1-2x)(1-3x)$$

If $x = 1/3 \Rightarrow \boxed{B=-4}$

$x = 1/4 \Rightarrow \frac{1}{4} + \frac{1}{16} = C(1-\frac{2}{4})(1-\frac{3}{4})$

$$\frac{4+1}{16} = C(\tfrac{1}{2})(\tfrac{1}{4})$$

$$\frac{5}{16} = \frac{1}{8}C \Rightarrow \boxed{C = 5/2}$$

If $x = 1/2 \Rightarrow \frac{1}{2} + \frac{1}{4} = A(1-\frac{3}{2})(1-\frac{4}{2})$

$$\frac{3}{\cancel{4}_2} = \frac{A}{\cancel{2}} \Rightarrow \boxed{A = 3/2}$$

$$A(x) = \frac{3/2}{1-2x} - \frac{4}{1-3x} + \frac{5/2}{1-4x}.$$

$$= \frac{3}{2}(1-2x)^{-1} - 4(1-3x)^{-1} + \frac{5}{2}(1-4x)^{-1}$$

$$= \frac{3}{2}\sum_{n=0}^{\infty} 2^n x^n - 4\sum_{n=0}^{\infty} 3^n x^n + \frac{5}{2}\sum_{n=0}^{\infty} 4^n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[\frac{3}{2}2^n - 4\,3^n + \frac{5}{2}4^n\right]x^n$$

$$a_n = \frac{3}{2}.2^n - 4.3^n + \frac{5}{2}.4^n.$$

Prob Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$ with $a_0 = 10$, $a_1 = 41$. (root. 2, 5)

$c_1 = 3$, $c_2 = 7$. $a_n = (3)2^n + 7.5^n$.

Method 3 Characterstic Roots

The method of solving homogeneous linear recurrence relation of degree "k" by the method of characterstic roots for this, we require the definition of the characterstic, Equation of a homogeneous linear recurrence relation.

Let $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0,\ n \geq k,\ c_k \neq 0$ — (1)

be a linear homogeneous recurrence relation of degree k.

Then the Equation $t^k + c_1 t^{k-1} + c_2 t^{k-2} + \cdots + c_k = 0$ — (2) is called the characterstic Equation of the given recurrence relation (1)

Degree of Equation (2) is k it is "k" roots.

Let $\alpha_1, \alpha_2, \ldots \alpha_k$ be the roots of the Equation (2) & the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called characterstic roots of the recurrence relation

Example :- The characterstic Equation

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t = 1, 2$ are characterstic roots.

Two types of characteristic roots

1) If the characteristic Equation of a linear homogeneous recurrence relation of degree "k" has "k" distinct roots say $\alpha_1, \alpha_2 \ldots \alpha_k$ then $a_n = c_1\alpha_1^n + c_2\alpha_2^n + \ldots + c_k\alpha_k^n$.

where $c_1, c_2, \ldots c_k$ are constants, is the General solution of the given recurrence relation.

2) If the characteristic Equation of a linear homogeneous recurrence relation of degree k has a root "2" repeated k times then

$$a_n = \left[D_1 + D_2 n + D_3 n^2 + \ldots + D_k n^{k-1}\right] 2^n.$$

$$\left[c_1 + c_2 n + c_3 n^2 + \ldots + c_k n^{k-1}\right] 2^n.$$

where $D_1, D_2, \ldots D_k$ are constants, is the General solution of the given recurrence relation.

$$a_n = \left[D_1 + D_2 n + D_3 n^2 + \ldots + D_k n^{k-1}\right] 2^n$$

where $D_1, D_2, \ldots D_k$ are constants is the General solution of the given recurrence relation.

1. Solve $a_n - 3a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$

The characterstic Equation is

$$t^2 - 3t + 2 = 0$$
$$(t-1)(t-2) = 0$$
$$t = 1, 2$$

General solution is $a_n = c_1(1)^n + c_2(2)^n$.
$= c_1 + c_2 2^n$.

2. Solve $a_n - 3a_{n-1} - 4a_{n-2} = 0$ $n \geq 2$

The Characterstic Equation

$$t^2 - 3t - 4 = 0$$
$$(t-4)(t+1) = 0$$
$$t = 4, -1$$

General solution is $a_n = c_1(-1)^n + c_2(4)^n$.

3. Solve $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $n \geq 2$

The characterstic Equation is

$$t^2 - 7t + 12 = 0$$
$$(t-3)(t-4) = 0$$
$$t = 3, 4$$

General solution is $a_n = c_1 3^n + c_2 4^n$.

4) Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$

Characteristic roots is $t^2 - 6t + 9 = 0$

$$(t-3)^2 = 0$$

$$t = 3, 3$$

General solution is $a_n = (D_1 + D_2 n)3^n$.

or $(c_1 + c_2 n)3^n$.

5) Solve $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$

characteristic Equation is $t^3 - 3t^2 + 3t - 1 = 0$

$$(t-1)^3 = 0$$

$$t = 1, 1, 1$$

General solution is $a_n = (c_1 + c_2 n + c_3 n^2)(1)^n$

$=$

Case (i) If $\alpha_1, \alpha_2, \ldots \alpha_k$ are roots of the characteristic Equation. Such that $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_k$. then solution is

$a_n = c_1\alpha_1^n + c_2\alpha_n^n + \cdots + c_k\alpha_k^n$.

Case 2 :- If two roots are repeated i.e $\alpha_1 = \alpha_2$ then solution is $(c_1 + c_2 n)\alpha_1^n$

Case 3: If two roots are imaginary i.e $\alpha_1 = \alpha + i\beta$, $\alpha_2 = \alpha - i\beta$

then $a_n = r^n[c_1\cos n\theta + c_2\sin n\theta]$

Case 4. Suppose 3 roots are repeated then solution is

$$a_n = (c_1 + c_2 n + c_3 n^2)\alpha^n.$$

Case 5: If two Complex roots are repeated

$\alpha_1 = \alpha_2 = \alpha + i\beta, \quad \alpha_3 = \alpha_4 = \alpha - i\beta$

then solution is $a_n = r^n[(c_1 + c_2 n)\cos n\theta + (c_3 + c_4 n)\sin n\theta]$

Problem Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0$ where $a_0 = 2$, $a_1 = 5$

$$t^2 - 5t + 6 = 0$$

$$(t-2)(t-3) = 0$$

$$t = 2, 3.$$

$$a_n = c_1 2^n + c_2 3^n \quad \text{— (1)}$$

$a_0 = 2;\ a_1 = 5$

Put $n=0$ $\quad a_0 = c_1 2^0 + c_2 3^0.$

$$\boxed{2 = c_1 + c_2} \quad \text{— (2)}$$

Put $n=1$ in eq (1)

$$a_1 = c_1 2^1 + c_2 3^1 \quad [a_1 = 5]$$

$$\boxed{5 = 2c_1 + 3c_2} \quad \text{— (3)}$$

Solve (2) & (3)

$$\boxed{c_1 = 1} \quad \boxed{c_2 = 1}$$

$$a_n = 2^n + 3^n.$$

Prob 2. Solve $a_n - 9a_{n-1} + 27a_{n-2} - 27a_{n-3} = 0$

The characterstic Equations is

$f(t) = t^3 - 9t^2 + 27t - 27 = 0$

$f(3) = 108 - 108$

$$\begin{array}{r|rrrr} 3 & 1 & -9 & 27 & -27 \\ & 0 & 3 & -18 & 27 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

$(t-3)$ is a factor of $f(t)$

$(t-3)(t^2 - 6t + 9) = 0$

$(t-3)(t-3)^2 = 0$

$t = 3, 3, 3$

General solution is $a_n = (D + D_2 n + D_3 n^2) 3^n$.

Prob 3 Solve the recurrence relation $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$ with the initial conditions $a_0 = 1, a_1 = 4, a_2 = 8$.

The characterstic Equation is $t^3 - 7t^2 + 16t - 12 = 0$

$F(t) = t^3 - 7t^2 + 16t - 12$.

$F(1) = -2 \neq 0$

$F(-1) = -36 \neq 0$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$(t-2)$ is one factor of $f(t)$

$f(t) = (t-2)(t^2 - 5t + 6)$

$= (t-2)(t-2)(t-3)$

$= (t-2)^2(t-3)$

$t = 2, 2, 3$.

$c_1 + c_3 = 1$

$2c_1 + 2c_2 + 3c_3 = 4$

$4c_1 + 8c_2 + 9c_3 = 8$

$c_1 = 5, c_2 = 3, c_3 = -4$.

General solution is $c_1 2^n + c_2 n 2^n + c_3 3^n$

$5 2^n + 3n 2^n - 4 3^n.$

Prob Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$

Characteristic ~~roots~~ Equation is $t^2 - 6t + 9 = 0$

$(t-3)^2 = 0$

$t = 3, 3$

General Solution is $a_n = (c_1 + c_2 n) 3^n$.

Prob Solve the following recurrence relations using the characteristic roots.

$a_n + a_{n-1} - 5a_{n-2} + 3a_{n-3} = 0$ where $a_0 = 0, a_1 = 1, a_2 = 2$

Characteristic Equation is $t^3 + t^2 - 5t + 3 = 0$

$(t-1)$ is a factor of $f(t)$

$$\begin{array}{c|cccc} 1 & 1 & 1 & -5 & 3 \\ & 0 & 1 & 2 & -3 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

$(t-1)(t^2 + 2t - 3) = 0$

$(t-1)(t+3)(t-1) = 0$

$t = 1, 1, -3.$

$a_n = (c_1 + c_2 n)(1)^n + c_3(-3)^n$

Put $n=0 \Rightarrow a_0 = c_1 + c_3 = 0$

$n=1 \Rightarrow a_1 = c_1 + c_2 - 3c_3.$

$n=2 \Rightarrow a_2 = c_1 + 2c_2 + 9c_3.$

$c_1 = 0, \ c_3 = 0, \ c_2 = 1.$

General solution is $a_n = [0 + 1(n)](1)^n + 0.$

$= n.$